Mathematics 181 – **D** **Fall 2006**

Exam 2

October 3, 2006

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Name

Technology used:____ Directions:

- Be sure to include in-line citations every time you use technology.
- Include a careful sketch of any graph obtained by technology in solving a problem.
- **Only write on one side of each page.**
- **When given a choice, specify which problem(s) you wish graded.**

The Problems

1. (15 points) Do **one** (1) of the following.

   (a) Find the area of the region bounded by the graphs of $x = y^2$ and $x = -2y^2 + 3$.

   i. Solving $y^2 = -2y^2 + 3$ we see the graphs intersect when $y = -1$ and $y = +1$ (the points $(1, -1)$ and $(1, 1)$).

   ii. Using horizontal rectangles the area is $\int_{-1}^{1} \left[\left(-2y^2 + 3\right) - y^2\right] \, dy = \int_{-1}^{1} \left[-3y^2 + 3\right] \, dy = -y^3 + 3y\Big]_{-1}^{1} = 4$

   (b) Find the area of the region in the first quadrant enclosed by the curves $y = \cos\left(\frac{\pi x}{2}\right)$ and $y = 1 - x^2$.

   i. Graphing we see there are only two points of intersection $(0, 0)$ and $(1, 1)$ and that the parabola graphs above the trigonometric function.

   ii. So the area is $\int_0^1 \left(1 - x^2\right) \, dx - \int_0^1 \cos\left(\frac{\pi x}{2}\right) \, dx$

   iii. The first inegral is $x - \frac{1}{3}x^3\Big]_0^1 = \frac{2}{3}$

   iv. For the second integral we use a substitution: $u = \frac{\pi}{2}x$ so that $du = \frac{\pi}{2}dx$ and $dx = \frac{2}{\pi}du$.In addition, $x = 0$ gives $u = 0$ and $x = 1$ gives $u = \frac{\pi}{2}$.

   v. The second integral is now $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(u) \, du = \frac{2}{\pi}\sin(u)\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi}$

   vi. The total area is the difference of the two integrals $A = \frac{2}{3} - \frac{2}{\pi}$

2. (15 points) Do **one** (1) of the following.

   (a) Evaluate

   $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos(\theta) \, d\theta}{1 + (\sin(\theta))^2}$$

   i. Using the substitution $u = \sin(\theta)$ we have $du = \cos(\theta) \, d\theta$ and new limits of integration $u = -1$ and $u = 1$.

   ii. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\cos(\theta)\,d\theta}{1+(\sin(\theta))^2} = 2\int_{-1}^{1} \frac{du}{1+u^2} = 2\arctan(u)]_{-1}^{1} = 2\left(\frac{\pi}{4}\right) - 2\left(-\frac{\pi}{4}\right) = \pi$

   (b) Solve the initial value problem $\frac{ds}{dt} = 8\sin^2\left(t + \frac{\pi}{12}\right)$, $s(0) = 8$.

i. We need a function whose derivative is $8\sin^2\left(t+\frac{\pi}{12}\right)$ and whose output is 8 when we input 0.

ii. We use a substitution to find an antiderivative of $8\sin^2\left(t+\frac{\pi}{12}\right)$: $u=t+\frac{\pi}{12}$ so $du=dt$.

iii. $\int 8\sin^2\left(t+\frac{\pi}{12}\right)\,dt = 8\int \sin^2(u)\,du = 8\left[\frac{u}{2}-\sin\left(\frac{2u}{4}\right)\right]+C$ so our function looks like

iv. $s(t)=8\left[\frac{t+\frac{\pi}{12}}{2}-\frac{\sin\left(2\left(t+\frac{\pi}{12}\right)\right)}{4}\right]+C.$

v. Plugging in $8=s(0)=8\left[0-\frac{\sin(\pi/6)}{4}\right]=-1$ we obtain

vi. $s(t)=8\left[\frac{t+\frac{\pi}{12}}{2}-\frac{\sin\left(2\left(t+\frac{\pi}{12}\right)\right)}{4}\right]-1=\left[4t+\frac{1}{3}\pi-2\sin\left(2t+\frac{1}{6}\pi\right)\right]-1$

3. (15 points) The base of a solid is the region in the xy -plane bounded by the graphs of the parabolas $y=2x^2$ and $y=5-3x^2$. Find the volume of the solid given that cross sections perpendicular to the x -axis are squares.

(a) Solving $2x^2=5-3x^2$ we get $x=-1$ and $x=1$.

(b) $A(x)=\left(5-3x^2-2x^2\right)^2=\left(5-5x^2\right)^2=25-50x^2+25x^4$

(c) $V=\int_{-1}^{1}A(x)\,dx=\int_{-1}^{1}\left(25-50x^2+25x^4\right)\,dx=25x-\frac{50}{3}x^3+5x^5\Big]_{-1}^{1}=\frac{80}{3}$

4. (15 points) Do both of the following. Use the Method of Slicing on one and the Method of Cylindrical Shells on the other.

(a) Set up, but **do not evaluate** a definite integral for the volume of the solid obtained when the region bounded by the graphs of the curves $y=\sqrt{2x}$ and $y=x$ is rotated about the line $y=-1$.

i. The two curves intersect when $\sqrt{2x}=x$ or $2x=x^2$ or $x=0,2$. So the points of intersection are $(0,0)$ and $(2,2)$. Solving for x the two equations tell us: $x=\frac{1}{2}y^2$ and $x=y$

ii. **Slicing:** The cross-sections are washers with large radius $R=1+\sqrt{2x}$ and small radius $r=1+x$ so the cross-sectional area function is $A(x)=\pi\left(1+\sqrt{2x}\right)^2-\pi(1+x)^2$. The volume is $V=\int_0^2\left[\pi\left(1+\sqrt{2x}\right)^2-\pi(1+x)^2\right]\,dx$

iii. **Cylindrical Shells:** The shell radius is $1+y$ and the shell height is $y-\frac{1}{2}y^2$. So the volume is $V=2\pi\int_0^2\left[(1+y)\left(y-\frac{1}{2}y^2\right)\right]\,dy$.

(b) Set up, but **do not evaluate** a definite integral for the volume of the solid obtained when the region bounded by the graphs of the curves $y=\sqrt{2x}$ and $y=x$ is rotated about the line $x=-1$.

i. The two curves intersect when $\sqrt{2x}=x$ or $2x=x^2$ or $x=0,2$. So the points of intersection are $(0,0)$ and $(2,2)$. Solving for x the two equations tell us: $x=\frac{1}{2}y^2$ and $x=y$.

ii. **Slicing:** The cross-sections are washers with large radius $1+y$ and small radius $1+\frac{1}{2}y^2$. So the volume is $V=\int_0^2\pi(1+y)^2-\pi\left(1+\frac{1}{2}y^2\right)^2\,dy$

iii. **Cylindrical Shells:** The shell radius is $(1+x)$ and the shell height is $\left(\sqrt{2x}-x\right)$. So the volume is $V = 2\pi \int_0^2 \left[(1+x)\left(\sqrt{2x}-x\right)\right] dx$.

5. (15 points) Find the total length of the graph of $f(x) = 1/3x^{3/2} - x^{1/2}$ from $x = 1$, to $x = 4$. [Hint: Δs is a perfect square.]

   (a) $f'(x) = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$ so $\sqrt{1+[f'(x)]^2} = \sqrt{1+\left(\frac{1}{2}x^{1/2}-\frac{1}{2}x^{-1/2}\right)^2} = \sqrt{1+\frac{1}{4}x-\frac{1}{2}+\frac{1}{4x}} = \sqrt{\frac{1}{4}x+\frac{1}{2}+\frac{1}{4x}} = \sqrt{\left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)^2} = \left|\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right| = \frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}$ because they term inside the absolute values is always positive for $1 \le x \le 4$.

   (b) Thus the length of the curve is $L = \int_1^4 ds = \int_1^4 \sqrt{1+[f'(x)]^2}\,dx = \int_1^4 \left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right) dx = \frac{1}{3}x^{3/2}+x^{1/2}\Big]_1^4 = \frac{10}{3}$

6. (10 points each) Do any **two** of the following.

   (a) Suppose that $F(x)$ is an antiderivative of $f(x) = \frac{\sin(x)}{x}$, $x > 0$. Express

   $$\int_1^3 \frac{\sin(2x)}{x}\,dx$$

   in terms of F.

   i. By the first part of the Fundamental Theorem of Calculus, $F'(x) = \frac{\sin(x)}{x}$ for any $x > 0$.

   ii. We make a substitution to $\int_1^3 \frac{\sin(2x)}{x}\,dx$ as follows: $u = 2x$, so $x = \frac{1}{2}u$ and $dx = \frac{1}{2}du$.

   iii. So $\int_1^3 \frac{\sin(2x)}{x}\,dx = \int_2^6 \frac{\sin(u)}{\frac{1}{2}u}\left(\frac{1}{2}\,du\right) = \int_2^6 \frac{\sin(u)}{u}\,du = F(x)]_2^6 = F(6) - F(2)$.

   (b) The disk enclosed by the circle $x^2+y^2 = 4$ is revolved about the y - axis to generate a solid ball. A hole of diameter 2 (radius 1) is then bored through the ball along the y -axis. Set up, but do not evaluate, definite integral(s) that give the remaining volume of this “cored” solid ball.

   i. **Slicing Method:** The cross-sections perpendicular to the y-axis are washers with large radius $R = \sqrt{4-y^2}$ and small radius 1. The hole touches the circle $x^2+y^2 = 4$ at the point $\left(1, \sqrt{3}\right)$.
   $V = \int_{-\sqrt{3}}^{\sqrt{3}} \left[\pi\left(\sqrt{4-y^2}\right)^2 - \pi(1)^2\right] dy$

   ii. **Cylindrical Shells:** The shell radius is x and the shell height is $2\sqrt{4-x^2}$.
   $V = 2\pi \int_1^2 x\left(2\sqrt{4-x^2}\right) dx$.

   (c) A solid is generated by rotating about the x - axis the region in the first quadrant between the the x - axis and the curve $y = f(x)$. The function f has the property that the volume, $V(x)$, generated by the part of the region above the interval $[0, x]$ is x^2 for every $x > 0$.Find the function $f(x)$.

   i. $V(x) = \int_0^x \pi[f(t)]^2\,dt = x^2$. Taking derivatives (using the FTC) we have $V'(x) = \pi[f(x)]^2 = 2x$.Solving for $f(x)$ gives us $f(x) = \sqrt{\frac{2x}{\pi}}$

   (d) Find the volume of the following “twisted solid”. A square of side length s lies in a plane perpendicular to line L. One vertex of the square lies on L. As this vertex moves a distance h along L, the square turns one revolution about L. Find the volume of the solid generated by this motion. Briefly explain your answer.

i. By Cavalieri's principle the volume depends only on the cross sections perpendicular to the axis. Since these are all squares of area $A(x) = s^2$ then, by the method of slicing, the total volume is $V = \int_0^h s^2\,dx = s^2x\big]_0^h = s^2h$.

(e) A solid sphere of radius R centered at the origin can be thought of as a nested collection of thin spherical shells.

i. Set up a Riemann sum approximating the volume of this solid sphere by adding up the volumes of the thin, nested spherical shells. [Use the fact that a spherical shell of radius x has surface area of $4\pi x^2$.]

A. Subdivide the interval $[0, R]$ into n subintervals using the partition $P = \{x_0, x_1, x_2, \cdots, x_n\}$

B. For $k = 1$ to n select a point c_k in the k'th subinterval.

C. The volume of the nested spherical shell with radius c_k is approximately equal to the surface area of the shell times the width of the k'th subinterval. Specifically, the volume of the single shell is about $4\pi (c_k)^2 \Delta x_k$

D. The associated Riemann Sum that approximates the total volume is

$$\sum_{k=1}^{n} 4\pi (c_k)^2 \Delta x_k$$

ii. Write the definite integral that is equal to the limit (as $\|P\| \to 0$) of this Riemann Sum.

A. Since the function $f(x) = 4\pi x^2$ is continuous everywhere we know the limit of the Riemann Sum exists and is equal to the definite integral $\int_0^R 4\pi x^2\,dx$.

iii. You may **not** use either the Method of Slicing or the Method of Cylindrical Shells.